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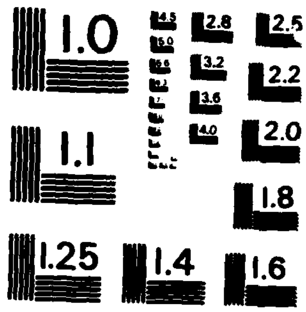

MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A computer program for the optimal economic design of an $\bar{X}$ control chart is presented. A single assignable cause system is assumed, where the mean time between process shifts is an exponentially distributed random variable. Given fixed and variable sampling costs, the costs of investigating action signals, the penalty cost of production in the out-of-control state, and other parameters describing process performance, the program finds the sample size, control limit width and interval between samples that minimize the expected total costs per unit time.

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COMPUTER PROGRAMS

Edited by Peter R. Nelson

Economic Design of an $\bar{X}$ Control Chart

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A computer program for the optimal economic design of an $\bar{X}$ control chart is presented. A single assignable cause system is assumed, where the mean time between process shifts is an exponentially distributed random variable. Given fixed and variable sampling costs, the costs of investigating action signals, the penalty cost of production in the out-of-control state, and other parameters describing process performance, the program finds the sample size, control limit width and interval between samples that minimize the expected total costs per unit time.

Introduction

CONTROL charts are widely used to maintain statistical control of a process. They are also used for analyzing process capability and estimating process parameters. To use a Shewhart control chart, we must select the same size n, the width of the control limits k, and the interval between samples h. We assume that k is a multiple of the standard deviation of the statistic plotted on the chart and that h is in hours. Selection of n, k and h is called the *design* of the control chart.

While control charts traditionally are designed with respect to statistical criteria, there has been much research devoted to the design of control charts using economic criteria. For a recent survey of this field, see Montgomery (1980). This note represents a computer program for the optimal economic design of an $\bar{X}$ control chart, based on the cost model of Duncan (1956).

The process is assumed to start in a state of statistical control with mean μ_0 and standard deviation σ. There is a single assignable cause that results in a shift in the process mean from μ_0 to $\mu_0 \pm \delta\sigma$, where δ is known. The time before the assignable cause occurs has an exponential distribution with parameter λ (thus λ^{-1} is the mean time in the in-control state). Samples of size n are taken every h hours and the sample mean is plotted on an $\bar{X}$ control chart with center line μ_0 and control limits $\mu_0 \pm k\sigma/\sqrt{n}$. If one point falls outside the control limits, a search for the assignable cause is made. The process continues in operation during this search and the repair cost is not charged to the process operating cost.

If one defines

a_1 = the fixed cost of sampling

a_2 = the variable cost of sampling

a_3 = the cost of finding an assignable cause

a_3' = the cost of investigating a false alarm

a_4 = the hourly penalty cost for operating in the out-of-control state

α = the type I error rate

$1 - \beta$ = the power of the chart

g = the time required to sample one item and interpret the results

D = the time required to find the assignable cause following an action signal

then Duncan's (1956) paper shows that under the above assumptions the expected cost per hour in-

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curred by the process is

$$E(L) = \frac{a_1 + a_2 n}{h} + \{a_4[h/(1-\beta) - \tau + gn + D] + a_3 + a_3' \alpha e^{-\lambda h}/(1 - e^{-\lambda h})\} \Big/ [1/\lambda + h/(1-\beta) - \tau + gn + D] \quad (1)$$

where

$$\tau = \frac{\int_{jh}^{(j+1)h} e^{-\lambda t}(t - jh)\, dt}{\int_{jh}^{(j+1)h} e^{-\lambda t}\lambda\, dt} = \frac{1 - (1 + \lambda h)e^{-\lambda h}}{\lambda(1 - e^{-\lambda h})} \quad (2)$$

is the expected time of occurrence of the assignable cause, given that it occurs between the jth and $(j+1)$st samples.

Some simplification of this cost function is possible. Duncan (1956) notes that

$$\tau \simeq h\left(\frac{1}{2} - \frac{\lambda h}{12}\right) \quad (3)$$

$$\alpha e^{-\lambda h}/(1 - e^{-\lambda h}) \simeq \alpha/\lambda h. \quad (4)$$

Substituting (3) and (4) into (1) and simplifying gives

$$E(L) = \frac{a_1 + a_2 n}{h} + \frac{\lambda B a_4 + \alpha a_3'/h + \lambda a_3}{1 + \lambda B} \quad (5)$$

where

$$B = [1/(1-\beta) - 1/2 + \lambda h/12]h + gn + D. \quad (6)$$

Note that $E(L)$ is a function of the control chart parameters n, h and k. Equation (5) may be minimized with respect to these parameters by direct search methods. Other optimization methods are discussed by Duncan (1956); Goel, Jain and Wu (1968); and Chiu and Wetherill (1974).

Program Description

Various numerical studies have indicated that the optimal control chart design is relatively insensitive to misspecification of the cost parameters but relatively sensitive to the magnitude of the shift δ. Furthermore, the magnitude of the shift primarily affects the optimal sample size n. For this reason, the program displays a number of control chart designs in the neighborhood of the optimum. The suboptimal designs use a different sample size than the optimal design but have been optimized with respect to n and k. This display is intended to give the analyst some feeling for the sensitivity of the cost surface. Based on this information, the analyst may elect to deviate somewhat from the optimal design, depending on his degree of confidence in the parameter estimates.

Equation (5) is optimized in two stages. Chiu and Wetherill (1974) note that by constraining the power of the chart $(1-\beta)$ to a specified value (say 0.90 or 0.95) the optimal n and k can be approximated by the solution to

$$\frac{z + k}{\phi(k)} = \frac{\delta^2 a_3'}{a_2 + \lambda a_4 g} \quad (7)$$

and

$$\delta\sqrt{n} - k = z \quad (8)$$

where $z = 1.2816$ if $1 - \beta = 0.90$ and $z = 1.6449$ if $1 - \beta = 0.95$ and $\phi(u)$ is the density function of a standard normal random variable. The program uses $z = 1.2816$ to solve (7) and (8). The resulting n, say n^*, from (8) is used to set lower and upper limits in the search for the optimal sample size.

The second phase of the optimization finds the optimal k and h for each value of n in the interval $\max(1, n^* - 10) \leq n \leq n^* + 10$. The control limit k is found using a three-stage line search starting with a coarse grid, followed by two successively finer meshes. At each cost function evaluation, the optimal h is computed using the following equation.

$$h = \left\{\frac{\alpha a_3' + a_1 + a_2 n}{\lambda a_4[1/(1-\beta) - 0.5]}\right\}^{1/2} \quad (9)$$

This approximation for the optimal h given n and k was suggested by Duncan (1956) and Chiu and Wetherill (1974). An exact closed form solution for the optimal h derived by Goel, Jain and Wu (1968) could be used instead of equation (9). In practice, however, this refinement seems unnecessary.

The values α and $1 - \beta$ are given by $\alpha = 2\Phi(-k)$ and $1 - \beta = \Phi(\delta\sqrt{n} - k) + \Phi(-\delta\sqrt{n} - k)$ where $\Phi(\cdot)$ is the cumulative standard normal distribution function, which is evaluated in the program using an approximation from Abramowitz and Stegun (1964, equation 26.2.17).

Program Operation

The user must supply the nine parameters a_1, a_2, a_3, a_3', a_4, λ, δ, g and D. The program calculates the

optimal control limit width k and sampling frequency h for several sample sizes and displays the corresponding value of the cost function, equation (6). The α risk (false alarm probability) and power $1 - \beta$ for each combination n, k and h are also provided. The optimal control chart design (n, k, h) is found by inspecting the output values of the cost function to find the minimum.

The input parameters a_1, a_2, a_3, a_3', a_4, λ, δ, g and D are entered in that order on a single data card using a 9F8.0 format.

Examples

Example One

Suppose that the fixed cost of sampling is $1.00 and the variable cost of sampling is estimated to be $0.10. It takes approximately one minute (0.0167 hours) to take and analyze each observation. The magnitude of the process shift is two standard deviations and process shifts occur according to an exponential distribution with a frequency of about one every twenty hours of operation. Thus $\lambda = 0.05$. It takes one hour to investigate an action signal following the process shift. The cost of investigating a false alarm is $50 and a true action signal costs $25 to investigate. The hourly penalty cost for operating in the out-of-control state is $100. The input for the program is the following.

1.0 0.1 25.0 50.0 100.0 0.05 2.0 0.0167 1.0

The output is shown in Output Listing 1. Note that the optimal design has $n = 5$, $k = 2.99$ and $h = 0.76$ hours, with a minimum cost of $10.38 per hour. The α risk for this control chart design is 0.0028 and the power of the chart is $1 - \beta = 0.9308$. Notice that there are several other designs that employ a sample size slightly different from the optimal value of $n = 5$ that are close to the optimal in terms of minimum cost.

OUTPUT LISTING 1. Computer Output for Example 1

Example Two

To illustrate the sensitivity of the control chart to the magnitude of the process shift, we will rerun example one with $\delta = 1$. The input for the program is now as follows.

1.0 0.1 25.0 50.0 100.0 0.05 1.0 0.0167 1.0

The computer output is given in Output Listing 2. Note that two designs now have the same minimum cost of $12.89 per hour: $n = 13$, $k = 2.59$, $h = 0.91$, and $n = 14$, $k = 2.61$, $h = 0.94$. The effect of decreasing the magnitude of the shift from 2σ to 1σ has been to increase greatly the sample size, reduce slightly the width of the control limits and increase the interval between samples.

OUTPUT LISTING 2. Computer Output for Example 2

The reader is reminded of two assumptions in the development and use of this economic model which are potentially critical. The first of these is that the process is allowed to continue in operation during searches for the assignable cause and the cost of repair is not charged against operating costs. A model that assumes the process is stopped during searches and which includes the cost of repair is in Montgomery (1980). Use of the wrong process model may result in a control chart design that is far from optimal. The second critical assumption is the use of the exponential distribution as a model for the time between process shifts. If the process failure mechanism does not have the "memoryless" property implied by the exponential distribution, the control chart designs that result from this program may be significantly in error. Further discussion of this assumption and of models not requiring the exponential distribution may be found in Montgomery (1980).

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Program Listing

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